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Note

On the existence of (k,l)-critical graphs

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Abstract

Let $W \subseteq V$ in a graph $G=(V,E)$ such that $W \cap X \neq \emptyset$ for each fragment X of G. Then G is defined to be W-locally (k,l)-critical if $\kappa(G-W')=k-|W'|$ holds for every $W' \subseteq W$ with $|W'| \leqslant l$. In this note we give a short proof for the following recent result of Su: every non-complete W-locally (k,l)-critical graph has $(2l+2)$ distinct ends and $|W| \geqslant 2l+2$. (This result implies that Slater's conjecture is true: there exist no (k,l)-critical graphs with $2l>k$, except K_{k+1}.)

Let $G=(V,E)$ be a graph with vertex-connectivity $\kappa(G)=k$. For a set $X \subseteq V$ of vertices $N_G(X)$ (or simply $N(X)$) denotes the set of *neighbours* of X, i.e., $N(X)=\{v \in V-X\colon uv \in E$ for some $u \in X\}$. Let $n(X)=|N(X)|$. A set $X \subset V$ with $|V-X| \geqslant k+1$ is a *fragment* if $n(X)=k$. If X is a fragment, $V-X-N(X)$ is a fragment, as well. A minimal (for inclusion) fragment is an *end.* A fragment of smallest cardinality is an *atom.* A separating set T of vertices in G is a *cut*, if $|T|=k$ holds. We say that an edge e in a k-connected graph G is *essential* (with respect to k-connectivity), if $\kappa(G-e)=k-1$. Let $W \subseteq V$ such that $W \cap X \neq \emptyset$ for each fragment X of G. Then G is defined to be *W-locally (k,l)-critical* if $\kappa(G-W')=k-|W'|$ holds for every $W' \subseteq W$ with $|W'| \leqslant l$. If G is W-locally (k,l)-critical with $W=V$, then G is said to be *(k,l)-critical.* It is easy to see that G is W-locally (k,l)-critical if and only if $\kappa(G)=k$ and for any $W' \subseteq W$ with $|W'| \leqslant l$ there exists a cut T of G with $W' \subseteq T$. A subset $S \subseteq V$ is a *fragment-cover* of G if $S \cap X \neq \emptyset$ for each fragment X of G. (Clearly S is a fragment-cover if and only if S is an end-cover.)

Our proof for Su's theorem [6] is based on a lemma (Lemma 2 below) due to Mader, and some ideas from [1]. For deducing Slater's conjecture, we shall need another well-known lemma of Mader (Lemma 1).

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Lemma 1 (Mader [2]). *Let A be an atom and T a cut in a graph G. Then either $A \cap T = \emptyset$, or $A \subset T$ and $2|A| \leqslant \kappa(G)$.*

Lemma 2 (Mader [3]). *Let C be a cycle in a k-connected graph G such that every edge of C is essential. Then there exists a vertex v in C with $d_G(v) = k$.*

Theorem 3 (Su [6]). *Let $G = (V, E)$ be a non-complete W-locally (k, l)-critical graph. Then G has at least $2l + 2$ distinct ends and $|W| \geqslant 2l + 2$.*

Proof. Let $S \subseteq W$ be a minimal fragment-cover of G. By the definition of W, such an S exists. Let $F' = \{uv\colon\ u, v \in S\}$ be the set of all edges (not necessarily in E) spanned by S. Now $G'' = (V, E \cup F')$ is clearly $(k+1)$-connected. Let $F \subseteq F'$ be a minimal set of edges for which $G' = (V, E \cup F)$ is $(k+1)$-connected. We claim that the graph (S, F) is a forest. For suppose some edges of F form a cycle C. The minimality of F implies that every edge of C is essential in G' (with respect to $(k+1)$-connectivity) and for every vertex v of C we have $d_{G'}(v) \geqslant k+2$ since $d_G(v) \geqslant k$ holds by the k-connectivity of G. This contradicts Lemma 2. Thus, since F forms a forest (a bipartite graph on $|S|$ vertices), the edges of F can be covered by a set $S' \subset S$ with $|S'| \leqslant \lfloor |S|/2 \rfloor$. Now G is W-locally (k, l)-critical, which implies that if $|S'| \leqslant l$, then S' is included in a cut T of G. In this case the cut T, for which there is no edge of F in $G - T$, would still be a separating set of size k in G', as well. Thus $|S'| \geqslant l + 1$ and $|W| \geqslant |S| \geqslant 2l + 2$ holds. Furthermore, since S is a minimal cover of the ends of G, for any $s \in S$ there exists an end X_s with $X_s \cap S = \{s\}$. Hence the family $\mathscr{X} = \{X_s\colon\ s \in S\}$ consists of at least $2l + 2$ distinct ends of G. □

Corollary 4. *If G is a non-complete W-locally (k, l)-critical graph and $2l \geqslant k$, then G has $2l + 2$ pairwise disjoint fragments.*

Proof. Let $\mathscr{X}$ be the same as in the previous proof. Assume that $X_s \cap X_t \neq \emptyset$ for two distinct vertices $s, t \in S$. Since $|S| \geqslant 2l + 2$ and $2l \geqslant k$, we have $|V - (X_s \cup X_t)| \geqslant k$, which implies $n(X_s \cup X_t) \geqslant k$ by the k-connectivity of G. Thus, the well-known submodular property of the function n yields $k + k = n(X_s) + n(X_t) \geqslant n(X_s \cap X_t) + n(X_s \cup X_t) \geqslant k + k$, which implies that $X_s \cap X_t$ is a fragment (see also [1, Lemma 1.2]), contradicting the minimality of the fragment X_s (and X_t). Thus the family $\mathscr{X}$ consists of at least $2l + 2$ pairwise disjoint ends. □

Since Theorem 3 implies Slater's conjecture (see [6, Corollary 3]), our proof yields a short proof for the latter, as well. For completeness, we prove this corollary here — in a slightly generalized form — following the same idea which was used by Mader in [4] where he deduced Slater's conjecture [5] from his conjecture, which stated that every non-complete (k, l)-critical graph G has at least $2l + 2$ pairwise disjoint fragments. (Note, that this latter conjecture was proven recently by Su [7], provided that G has more than $k(l + 2)$ vertices.)

Theorem 5. *There exist no non-complete W-locally (k,l)-critical graphs for $2l>k$.*

Proof. Suppose that G is a W-locally (k,l)-critical graph with $2l>k$. Let A be an atom of G and $H=G-A$. Let $w\in A\cap W$. First observe that $\kappa(H)=k-|A|$ since there exists a cut T in G which contains w and by Lemma 1 we get that $A\subset T$ also holds. Moreover, any fragment X of H is a fragment of G, thus $W_H=W-A$ is a fragment-cover of H. For any set $W'_H\subseteq W_H$ with size at most $l-1$ we have that $W'_H\cup\{w\}$ is included in a cut T of G, for which $A\subset T$ also holds by Lemma 1. Thus W'_H is included in the cut $T-A$ of H. These facts imply that H is a non-complete W_H-locally $(k-|A|,l-1)$-critical graph.

Clearly, $2(l-1)\geqslant k-|A|$, thus Corollary 4 yields that H has $2l$ pairwise disjoint ends $X_1,\dots,X_{2l}$. For each i in $1\leqslant i\leqslant 2l$, the set $N_H(X_i)\cup A$ is a cut in G, hence $X_i\cap N_G(A)\neq\emptyset$. This implies $k\geqslant 2l$, a contradiction. □

Note, that there exist (k,l)-critical graphs for every $k\geqslant 1$ and $l\leqslant k/2$. A classical example for a $(2l,l)$-critical graph is the graph S_l which can be constructed from K_{2l+2} by deleting $l+1$ independent edges.

References

[1] T. Jordán, On the optimal vertex-connectivity augmentation, J. Combin. Theory Ser. B. 63 (1995) 8–20.

[2] W. Mader, Eine Eigenschaft der Atome endlicher Graphen, Arch. Math. 22 (1971) 333–336.

[3] W. Mader, Ecken vom Grad n in minimalen n-fach zusammenhängenden Graphen, Arch. Math. 23 (1972) 219–224.

[4] W. Mader, On k-critically n-connected graphs, in: J.A. Bondy, U.S.R. Murty (Eds.), Progress in Graph Theory, Academic Press, New York, 1984, pp. 389–398.

[5] S. Maurer, P.J. Slater, On k-critical n-connected graphs, Discrete Math. 20 (1977) 255–262.

[6] J. Su, On locally k-critically n-connected graphs, Discrete Math. 120 (1993) 183–190.

[7] J. Su, Fragments in k-critical n-connected graphs, J. Graph Theory 20 (3) (1995) 287–295.